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LETTER TO THE EDITOR

The soliton content of classical Jackiw–Teitelboim gravity

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Abstract

It is pointed out that *every* generic—in a sense to be made precise in section 2—solution to an *arbitrary* equation describing pseudo-spherical surfaces (or, equivalently, an arbitrary equation which is the integrability condition of a $sl(2, \mathbf{R})$-valued linear problem) determines pseudo-Riemannian surfaces of constant scalar curvature, and therefore, classical solutions to the Jackiw–Teitelboim field equations for two-dimensional gravity. In particular, this observation explains why some standard soliton equations appear in this theory.

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1. Introduction

This letter is about the two-dimensional gravity model proposed in the early 1980s by Teitelboim [36, 37] and Jackiw [14, 15], partially motivated by string theory [27]. The Jackiw–Teitelboim (henceforth JT) model is one of the simplest 2D gravity theories [18] but a very interesting one: it possesses Hamiltonian formulations (see [21, 36, 37] and also [25, 26, 34, 38]) and its quantization has been studied from several points of view [8, 13, 25, 26, 34]; it can be cast as a gauge theory [8, 13, 34]; and it admits a black-hole solution [20]—a dimensional reduction of the Bañados–Teitelboim–Zanelli 3D black hole [2, 3]—which possesses interesting thermodynamical and quantum properties [4, 7, 19].

The classical solutions to the JT gravity model are pseudo-Riemannian surfaces of constant scalar curvature [36, 37] and, since two surfaces of the same constant scalar curvature are (locally) isometric, it is natural to consider them using special coordinate systems. For example, it is very natural to study this theory in the Liouville gauge [15, 21], as in conformal coordinates the fact that a surface has constant scalar curvature is equivalent to the conformal factor satisfying the Liouville equation. More recently, Gegenberg and Kunstatter [11, 12] have studied the JT gravity in the Euclidean sine-Gordon gauge and showed, in particular, that the black holes encountered in this theory can be understood in terms of sine-Gordon solitons; Martina, Pashaev and Soliani [22–24] have investigated the gauge formulation of the JT gravity and obtained a system of equations—and an associated bi-Hamiltonian hierarchy of equations—whose solutions determine solutions to the field equations of the Jackiw–Teitelboim theory and moreover, they have remarked that for special reductions of their system the KdV and mKdV hierarchies naturally appear; finally, Bracken [6] has considered a Chern–Simons-type action, re-derived the system of equations appearing in [22–24], and connected it to the continuous Heisenberg model and the nonlinear Schrödinger equation.

The goal of this letter is to offer a simple geometric explanation of why integrable equations appear in this context. It will be shown that *every* generic solution of equations which describe pseudo-spherical surfaces—a class of equations introduced by Chern and Tenenblat [9], preeminent members of which are the sine-Gordon, KdV, mKdV and Liouville equations —provides models of pseudo-Riemannian surfaces of constant scalar curvature, and therefore, classical solutions to the Jackiw–Teitelboim field equations.

2. Equations of pseudo-spherical type

Definition 1. *An arbitrary scalar differential equation* $\Xi(x, t, u, \ldots) = 0$ *for a real-valued function* $u(x, t)$ *is said to describe pseudo-spherical surfaces or to be of pseudo-spherical type if and only if there exist smooth functions* $f_{\alpha\beta}, \alpha = 1, 2, 3, \beta = 1, 2$*, depending on* x, t, u*, and a finite number of derivatives of* u *such that the 1-forms*

$$\omega^\alpha = f_{\alpha 1}\,\mathrm{d}x + f_{\alpha 2}\,\mathrm{d}t$$

satisfy the structure equations of a surface of constant Gaussian curvature equal to -1 *with metric* $(\omega^1)^2 + (\omega^2)^2$ *and connection 1-form* ω^3*, namely,*

$$\mathrm{d}\omega^1 = \omega^3 \wedge \omega^2, \qquad \mathrm{d}\omega^2 = \omega^1 \wedge \omega^3 \qquad \text{and} \qquad \mathrm{d}\omega^3 = \omega^1 \wedge \omega^2 \tag{1}$$

whenever $u(x, t)$ *is a solution of the equation* $\Xi(x, t, u, \ldots) = 0$.

The trivial case of all functions $f_{\alpha\beta}$ depending only on the independent variables x, t is excluded from the considerations below. Equations of pseudo-spherical type were considered for the first time by Chern and Tenenblat [9], motivated by Sasaki's observation [33] that equations which are the necessary and sufficient condition for the integrability of a linear problem of AKNS type (Ablowitz, Kaup, Newell and Segur [1]) do describe pseudo-spherical surfaces. Large classes of equations admitting this structure have been characterized [9, 16, 29, 30], and geometrical methods for constructing solutions, (generalized) Bäcklund transformations, and conservation laws for these equations have been developed by several researchers [9, 17, 31–33, 35].

The expression 'PSS equation' will be sometimes utilized instead of 'equation describing pseudo-spherical surfaces'. The interpretation of definition 1 in terms of intrinsic differential geometry of surfaces is based on the following genericity notions [32]:

Definition 2. *Let* $\Xi = 0$ *be a PSS equation with associated 1-forms* $\omega^\alpha, \alpha = 1, 2, 3$*. A solution* $u(x, t)$ *of* $\Xi = 0$ *will be called I-generic if* $(\omega^3 \wedge \omega^2)(u(x, t)) \neq 0$*; II-generic if* $(\omega^1 \wedge \omega^3)(u(x, t)) \neq 0$*; and III-generic if* $(\omega^1 \wedge \omega^2)(u(x, t)) \neq 0$.

Proposition 1. *Let* $\Xi = 0$ *be a PSS equation with associated 1-forms* $\omega^\alpha, \alpha = 1, 2, 3$*, and let* $u(x, t)$ *be a local solution to* $\Xi = 0$*. Then,*

(a) If $u(x, t)$ *is a I-generic solution, the 1-forms* $\sigma_1 = \omega^2(u(x, t))$ *and* $\sigma_2 = \omega^3(u(x, t))$ *determine a Lorentzian metric of constant Gaussian curvature* $K = -1$ *on the domain* S *of* $u(x, t)$*, with connection 1-form given by* $\sigma_{12} = \omega^1(u(x, t))$.

(b) *If* $u(x,t)$ *is a II-generic solution, the 1-forms* $\sigma_1 = \omega^1(u(x,t))$ *and* $\sigma_2 = -\omega^3(u(x,t))$ *determine a Lorentzian metric of constant Gaussian curvature* $K = -1$ *on the domain* S *of* $u(x,t)$*, with connection 1-form given by* $\sigma_{12} = \omega^2(u(x,t))$*.*

(c) *If* $u(x,t)$ *is a III-generic solution, the 1-forms* $\sigma_1 = \omega^1(u(x,t))$ *and* $\sigma_2 = \omega^2(u(x,t))$ *determine a Riemannian metric of constant Gaussian curvature* $K = -1$ *on the domain* S *of* $u(x,t)$*, with connection 1-form given by* $\sigma_{12} = \omega^3(u(x,t))$*.*

The invariance properties of equations (1) are what one would expect:

Proposition 2. *Let* $\Xi = 0$ *be a PSS equation with associated 1-forms* ω^α, $\alpha = 1, 2, 3$*, and let* ρ *be any smooth function depending on* x, t, u*, and a finite number of derivatives of* u*. Then, whenever* $u(x,t)$ *is a solution to* $\Xi = 0$*, equations (1) are invariant under the transformations*

$$\widehat{\omega}^1 = \omega^1 \cos\rho + \omega^2 \sin\rho, \qquad \widehat{\omega}^2 = -\omega^1 \sin\rho + \omega^2 \cos\rho, \qquad \widehat{\omega}^3 = \omega^3 + d\rho; \tag{2}$$

$$\widehat{\omega}^1 = \omega^1 \cosh\rho - \omega^3 \sinh\rho, \qquad \widehat{\omega}^2 = \omega^2 + d\rho, \qquad \widehat{\omega}^3 = -\omega^1 \sinh\rho + \omega^3 \cosh\rho; \tag{3}$$

$$\widehat{\omega}^1 = \omega^1 + d\rho, \qquad \widehat{\omega}^2 = \omega^2 \cosh\rho + \omega^3 \sinh\rho, \qquad \widehat{\omega}^3 = \omega^2 \sinh\rho + \omega^3 \cosh\rho. \tag{4}$$

If $u(x,t)$ is III-generic, the pull-back of (2) by $u(x,t)$ is simply the transformation induced on the 1-forms $\omega^\alpha(u(x,t))$ by a rotation of the moving orthonormal frame dual to the coframe $\{\omega^1(u(x,t)), \omega^2(u(x,t))\}$, if $u(x,t)$ is II-generic the pull-back of (3) by $u(x,t)$ corresponds to a Lorentz boost of the moving frame dual to the coframe $\{\omega^1(u(x,t)), -\omega^3(u(x,t))\}$, and if $u(x,t)$ is I-generic the pull-back of (4) by $u(x,t)$ corresponds to a Lorentz boost of the frame dual to $\{\omega^2(u(x,t)), \omega^3(u(x,t))\}$.

Proposition 3. *Let* $\Xi = 0$ *be an equation of pseudo-spherical type with associated 1-forms* ω^α, $\alpha = 1, 2, 3$*. The equation* $\Xi = 0$ *is the integrability condition of the* $sl(2, \mathbf{R})$*-valued linear problem* $\mathrm{d}v = \Omega v$*, in which* Ω *is the 1-form*

$$\Omega = \frac{1}{2}\begin{pmatrix} \omega^2 & \omega^1 - \omega^3 \\ \omega^1 + \omega^3 & -\omega^2 \end{pmatrix}, \tag{5}$$

i.e. $\mathrm{d}\Omega = \Omega \wedge \Omega$ *whenever* $u(x,t)$ *is a local solution of* $\Xi = 0$*. Conversely, each* $sl(2, \mathbf{R})$*-valued 1-form* Ω *satisfying the zero curvature condition* $\mathrm{d}\Omega - \Omega \wedge \Omega = 0$ *on solutions to* $\Xi = 0$ *can be used, as in (5), to construct 1-forms* ω^α, $\alpha = 1, 2, 3$*, satisfying the structure equations (1) on solutions to* $\Xi = 0$*.*

In the terminology of Crampin, Pirani and Robinson [10] the 1-form $\Omega(u(x,t))$ determines a soliton connection. The choice (5) is motivated by the relation between the 1-forms ω^α associated with a PSS equation $\Xi = 0$, and the Maurer–Cartan structure equations of $SL(2, \mathbf{R})$, see [32]. This close connection between equations of pseudo-spherical type and linear problems was observed already by Sasaki [33].

3. Equations describing surfaces of constant curvature

Consider now a (pseudo)Riemannian manifold M of index i_M and constant sectional curvature K [5, 28, 35]. Assume that capital indices I, J take the values $1, 2, \ldots, N$, in which $N = \dim(M)$. Let e_I be an orthonormal moving frame on M, so that

$$\langle e_I, e_J \rangle = \epsilon_I \delta_{IJ}, \tag{6}$$

in which $\epsilon_I = 1$ for all I except for i_M indices for which $\epsilon_I = -1$. Let ω^I be the corresponding dual 1-forms, and define connection 1-forms ω_{IJ} as

$$\mathrm{d}e_I = \sum_{J=1}^{N} \epsilon_J \omega_{IJ} e_J. \tag{7}$$

These forms satisfy the metric compatibility constraint $\omega_{IJ} + \omega_{JI} = 0$. The structure equations of M are [5, 35]

$$\mathrm{d}\omega^I = \sum_{J=1}^{N} \epsilon_I \omega^J \wedge \omega_{JI}, \qquad \mathrm{d}\omega_{IJ} = \sum_{L=1}^{N} \epsilon_L \omega_{IL} \wedge \omega_{LJ} + \Omega_{IJ}, \tag{8}$$

in which $\Omega_{IJ} = -K\epsilon_I\epsilon_J\omega^I \wedge \omega^J$ (no summation). Taking $N = 2$, one obtains

$$\mathrm{d}\omega^1 = \epsilon_1\omega_{12} \wedge \omega^2, \qquad \mathrm{d}\omega^2 = \epsilon_2\omega^1 \wedge \omega_{12} \qquad \text{and} \qquad \mathrm{d}\omega_{12} = -K\epsilon_1\epsilon_2\omega^1 \wedge \omega^2, \tag{9}$$

which of course reduce to equations (1) for $\epsilon_1 = \epsilon_2 = 1$ and $\omega^3 = \omega_{12}$. Now restrict to the Lorentzian case. The structure equations for a two-dimensional pseudo-Riemannian manifold M with metric $\mathrm{d}s^2 = (\omega^1)^2 - (\omega^2)^2$ of signature $(1, -1)$ and connection 1-form ω_{12}, are

$$\mathrm{d}\omega^1 = \omega_{12} \wedge \omega^2, \qquad \mathrm{d}\omega^2 = -\omega^1 \wedge \omega_{12} \qquad \text{and} \qquad \mathrm{d}\omega_{12} = K\omega^1 \wedge \omega^2, \tag{10}$$

in which K is the Gaussian curvature of M. Recalling that the scalar curvature R satisfies $R = 2K$ [28], one arrives to the following definition:

Definition 3. *A differential equation $\Xi(x, t, u, \ldots) = 0$ describes Lorentzian surfaces of constant scalar curvature Λ if and only if there exist functions $f_{\alpha\beta}$, $\alpha = 1, 2, 3$, $\beta = 1, 2$, depending on x, t, u, and a finite number of derivatives of u, such that the 1-forms*

$$\omega^1 = f_{11}\,\mathrm{d}x + f_{12}\,\mathrm{d}t, \qquad \omega^2 = f_{21}\,\mathrm{d}x + f_{22}\,\mathrm{d}t, \qquad \omega_{12} = f_{31}\,\mathrm{d}x + f_{32}\,\mathrm{d}t, \tag{11}$$

satisfy the structure equations (10) with $K = \Lambda/2$ whenever $u(x, t)$ is a solution to $\Xi = 0$.

Definition 3 is a natural analogue of the definition of a PSS equation, of course. In fact, these two classes of equations coincide:

Proposition 4. *The 1-forms $\omega^1, \omega^2, \omega_{12}$ satisfy the structure equations (10) with $K = \Lambda/2$ if and only if the $sl(2, \mathbf{R})$-valued 1-form*

$$\Omega = \frac{1}{2}\begin{pmatrix} \omega_{12} & 2c(\omega^2 + \omega^1) \\ -2e(\omega^2 - \omega^1) & -\omega_{12} \end{pmatrix} \tag{12}$$

in which c and e are numbers, such that $ce = -\Lambda/8$, satisfies $\mathrm{d}\Omega - \Omega \wedge \Omega = 0$.

Proposition 5. *Let σ_α be three 1-forms satisfying the structure equations of a pseudo-spherical surface*

$$\mathrm{d}\sigma_1 = \sigma_3 \wedge \sigma_2, \qquad \mathrm{d}\sigma_2 = \sigma_1 \wedge \sigma_3, \qquad \mathrm{d}\sigma_3 = \sigma_1 \wedge \sigma_2. \tag{13}$$

Then, the 1-forms

$$\omega^1 = -\frac{2}{\Lambda}[(e + c)\sigma_1 - (e - c)\sigma_3] \tag{14}$$

$$\omega^2 = -\frac{2}{\Lambda}[(e - c)\sigma_1 - (e + c)\sigma_3] \tag{15}$$

$$\omega_{12} = \sigma_2, \tag{16}$$

in which $ce = -\Lambda/8$*, satisfy the structure equations (10). Conversely, if* ω^1, ω^2 *and* ω_{12} *satisfy (10), then the 1-forms*

$$\sigma_1 = (c-e)\omega^2 + (c+e)\omega^1, \tag{17}$$

$$\sigma_2 = \omega_{12}, \tag{18}$$

$$\sigma_3 = -(e+c)\omega^2 + (e-c)\omega^1, \tag{19}$$

in which $ec = -\Lambda/8$*, satisfy the structure equations (13).*

The proof of proposition 4 is a simple computation. One then obtains proposition 5 by comparing the matrix-valued 1-form (12) with the matrix-valued 1-form appearing in section 2. Thus, every solution of an equation describing pseudo-spherical surfaces (in particular, any equation solvable by AKNS-inverse scattering techniques [1, 33]) describes Lorentzian surfaces of constant scalar curvature.

Corollary 1. *Let* $\Xi = 0$ *be a PSS equation with associated 1-forms* $\sigma_\alpha, \alpha = 1, 2, 3$*. Then, any II-generic solution to* $\Xi = 0$ *determines a Lorentzian metric of constant scalar curvature, and therefore a classical solution for the Jackiw–Teitelboim equation of motion*

$$R = \Lambda.$$

The proof of this corollary is straightforward: since $\Xi = 0$ is a PSS equation with associated one-forms σ_α, the structure equations (13) are satisfied on solutions. Thus one can define 1-forms as in (14)–(16), and conclude that $\Xi = 0$ describes Lorentzian surfaces of constant scalar curvature Λ. In order to show that (14) and (15) determine a moving coframe (and therefore a nondegenerate Lorentzian metric of constant scalar curvature) one needs to check the independence condition $\omega^1 \wedge \omega^2 \neq 0$. A short computation shows that $\omega^1 \wedge \omega^2 = (8/\Lambda^2)[e^2 + c^2]\sigma_1 \wedge \sigma_3$, which is not zero if one considers II-generic solutions to $\Xi = 0$.

The dilaton field which appears in the gauge description of the theory [8, 13] can also be interpreted in the context of PSS equations: one can see, following Chamseddine and Wyler [8], that it corresponds to a Lie algebra-valued scalar which is covariantly constant with respect to the connection 1-form (5). This observation has been made already (in the special case of the Euclidean sine-Gordon equation) by Gegenberg and Kunstatter [11, 12].

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